

Assignment 12

This homework is due *Tuesday* Dec 2.

There are total 35 points in this assignment. 35 points is considered 100%. If you go over 31 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers sections 6.2, 6.4 in Bartle–Sherbert.

- (1) [3pt] (6.2.3c) Find the points of relative extrema of the function $f(x) = x|x^2 - 12|$ for $-2 \leq x \leq 3$.
- (2) [2pt] (6.2.6) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (Hint: Apply the Mean Value theorem to \sin on the interval $[x, y]$.)
- (3) [2pt] (6.2.17) Let f, g be differentiable on \mathbb{R} and suppose that $f(0) = g(0)$, and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$. (Hint: Apply the Mean Value Theorem to $f - g$ on $[0, x]$.)
- (4) For a given function f and a point x_0 , find Taylor's polynomials $P_2(x)$, $P_5(x)$, $P_{2013}(x)$ of $f(x)$ at x_0 .
 - (a) [2pt] $f(x) = \sin x$ at $x_0 = \pi/2$. Compare to \cos at 0.
 - (b) [2pt] $f(x) = \cos x$ at $x_0 = -\pi/2$. Compare to \sin at 0.
 - (c) [2pt] $f(x) = x^3$ at $x_0 = 2$. Compare $P_3(x), P_5(x), P_{2013}(x)$ to $f(x)$.
 - (d) [2pt] $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.
 - (e) [2pt] $f(x) = \frac{1}{x}$ at $x_0 = 1$. Compare to the previous item. (You can take for granted that $(\sin x)' = \cos x, (\cos x)' = -\sin x$.)

- (5) [3pt] (Part of exercise 6.4.7) If $x > 0$, show that

$$\left| \sqrt[3]{1+x} - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2 \right) \right| \leq \frac{5}{81}x^3.$$

(Hint: Apply Taylor's Theorem to $f(x) = \sqrt[3]{1+x}$ with $n = 2$.)

- (6) (a) [2pt] Suppose $A \in \mathbb{R}$. Show that $\lim_{n \rightarrow \infty} \frac{A^n}{n!} = 0$.
 Hint: take tail of this sequence that starts with $m > 2|A|$ and represent

$$\frac{A^n}{n!} = \frac{A^m}{m!} \cdot \frac{A^{n-m}}{(m+1) \cdots n}.$$
- (b) [3pt] (6.4.8) If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \rightarrow \infty$, for each fixed x_0 and x .
- (c) [3pt] (6.4.9) If $g(x) = \cos x$, show that the remainder term in Taylor's Theorem converges to zero as $n \rightarrow \infty$, for each fixed x_0 and x .

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- (7) (6.4.14) Determine whether or not $x = 0$ is a point of relative extremum of the following functions:
- (a) [2pt] $f(x) = x^3 + 2$,
 - (b) [2pt] $f(x) = \cos x - 1 + \frac{1}{2}x^2$.
- (8) (3pt) Apply Newton's Method to find recursive relation for approximating $\sqrt[3]{2}$. Find an interval $I^* \subseteq I = [1, 2]$ s.t. convergence of a sequence given by the above relation and $x_1 \in I^*$ is guaranteed.